**Separable topological Space of hereditary**

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**Abstract :** A property of a topological space is termed hereditary if and only if every subspace of a space with the property also has the property. The purpose of this article is to prove that the topological property of separable space is hereditary.

**Key words :** topological space, separable space, hereditary

**Definition 1.** Let (*X*, Ω) be a topological space. Then (*X*, Ω) is a second axiom space if and only if there exists a countable base for Ω. This is known as the second axiom of countability.

**Axiom 1.** Let (*X*, Ω) be a second axiom space and *Z* ⊆ *X*. Then (*Z*, *Z* ∩ Ω) is a second axiom space.

**Definition 1.** Let (*X*, Ω) be a topological space and *E* ⊆ *X*. Then *E* is dense in *X* if and only if *κE* = *X*.

**Definition 2.** Let (*X*, Ω) be a topological space. Then (*X*, Ω) is separable if and only if there exists a countable dense subset of *X*.

We will show that the property of being separable is not hereditary by showing that every topological space is a subspace of a separable topological space. [2, p. 84]

**Theorem 1.** Let (*X*, Ω) be a topological space (in particular a non-separable space). Let ∞be a point such that ∞ ∉ *X*. Then *X*\* = *X* ∪ (∞) with the topology



is a separable topological space and (*X*, Ω) is a subspace.

**Proof:** First we will show that (*X\**, Ω\*) is a topological space.

(i) Let *λ*\* ⊆ Ω\*. If *λ*\* = φ then



Assume *λ*\* ≠ φ and *λ =* {*o* : *o* ∈ Ω such that *o* ∪ (∞) = *o*\* for *o*\* ∈ *λ*\*. Then  since *.*

(ii) Now suppose *λ*\* ⊆ Ω\* with *λ*\* finite and . If *λ*\* ≠ φ then



Otherwise



since



as *λ*\* finite implies *λ* is finite also. Therefore (*X*\*, Ω\*) is a topological space.

Now we will show that (*X\**, Ω\*) is separable. Let *o*\* ∈ Ω\*. Then (∞) ∈ *o*\*, since  where .

Let *x* ∈ *X*\*. Then for every *o*\* such that *x* ∈ *o*\* we have



and *x* ∈ *κ* whence *X*\* ⊆ *κ*(∞). Thus *κ*(∞) = *X*\*, and {∞} is a countable dense subset of *X*. Therefore (*X*, Ω) is separable.

Clearly (*X*, Ω) is a subspace of (*X*\*, Ω\*), since



and



**Theorem 2.** [3, p. 60) In every second axiom topological space separability is hereditary.

**Proof.** Since each sub space of a second axiom space is also a second axiom space, we need to show that every second axiom space is separable. Now (*X*, Ω) second axiom implies that there is a countable base *β* for Ω. Thus for every *o* ∈ Ω there exists a countable family -*β*\* ⊆ *β* such that 

Let *β* = {*Bn* : *n* is a positive integer} . Choose x*n*∈ *Bn*.

We may assume *n* ≥ 1 implies that *Bn*≠ *φ*. Put



Clearly *E* is a countable subset of *X*.

Now let *x* ∈ *X* and *βx* ∈ {*B* : *B* ∈ *β*} and *x* ∈ *β*}. Here *βx* is a base at *x* and *B* ∈ *βx* impliesthat there exists an *n* such that *B* = *Bn*, whence



so that **.** Thus *x* ∈ *κE,* and  Therefore *κE* = *X*, and *E* is a countable dense subset of *X*, whence (*X*, Ω) is separable.

**Theorem 3.** Every separable metric space is hereditarily separable.

**Proof.** This follows immediately from theorem 3 and the fact that a metric space is separable if and only if it is second axiom. [1, p. 121]

**References**

[1] Gaal, S. A. (1964). Point Set Topology. New York: Academic Press Inc.

[2] Pervin, W. J. (1964). Foundations of General Topology. New York: Academic Press Inc.

[3] Sierpinski, W. (1952). General Topology, trans. C. Cecilia Krieger. Toronto: University of Toronto Press.